OFFICE OF THE REGISTRAR OF BOOKS
10 JUL 1914
MADRAS

THE JOURNAL

OF THE

Indian Mathematical Society.

Vol. VI.] APRIL 1914. [No. 2.

PROGRESS REPORT.

The following gentlemen have been elected as members of the Society :—

(1) Mr. L. N. Subrahmaniam, M.A., L.T, Acting Professor of Mathematics, S. P. G. College, Trichinopoly;

(2) Mr. R. Mahadevan, B.A., Extra Assistant Supt., Bharati Vilas, Basvangudi, Bangalore; and

(3) Mr. R. Littlehailes, M.A., Professor of Mathematics, Presidency College, Madras.

2. The following books have been received from the publishers :—

(1) *Elementary Statics*—by Dr. R. S. Heath, Clarendon Press, Oxford, 1913, 4/6;

(2) *A General Course of Pure Mathematics*—by A. L. Bowley, Clarendon Press, Oxford, 1913, 7/6;

(3) *Cours de Mecanique, Tome I*—by Leon LeCornu, Gauthier-Villars, Paris, 1914, 18 frs.;

(4) *Introduction Geometrique* a Quelques Theories Physiques-by Emile Borel, Gauthier-Villars, Paris, 1914, 5 frs.;

(5) *Les Recents Progres du Systeme Metrique* (being the report presented to the Fifth General Conference for weights and measures)—by Ch. Ed. Guillaume, Gauthier-Vil. Paris, 1914, 3 frs.

(6) *Annuaire pour L'an 1914*—publie par de Bureau des Longitudes, (avec des Notices Scientifiques), Gauthier, Vil., Paris, 1. 5 frs.

(7) *Projective Geometry*—by Hatton, Cambridge University Press, 1913, 10/6;

(8) *Exercises in Mathematics*—by D. B. Mair, Macmillan & Co., 1914, 4/6 ;

(9) *First Year Course in Mathematics*—by K. J. Sanjana, and J. Cooper, Bombay, 1914, Rs. 1-12-0 ;

(10) *Panjab University Calendar*—for 1913-1914.

(11) *A Binary Canon*—showing the Residues of Powers of 2 for divisors under 1C00, and indices to Residences—compiled by Lt. Col. A. Cunningham, (under the auspices of the British Association Committee), London 1900.

3. The Audited Balance sheet for 1913, and the Budget for 1914 are published herewith for the information of the General Body.

POONA,
31st March 1914.

D. D. KAPADIA,
Hony. Joint Secretary.

THE INDIAN MATHEMATICAL SOCIETY.

Balance Sheet for 1913.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
To Balance from 1912.	240	3	7	By Books and Journals	319	4	0
,, Subscriptions current	1247	8	0	,, Library charges ...	314	0	0
,, do. arrears.	508	8	0	,, Journal Printing...	533	7	0
,, do. life ...	150	0	0	,, Ordinary working expenses... ...	234	11	10
,, do. Journal	215	7	0	,, Balance to 1914 ...	970	1	9
,, Miscellaneous ...	9	14	0				
TOTAL Rs....	2,371	8	7	TOTAL Rs....	2,371	8	7

MADRAS,
7th February 1914.

(Sd.) C. POLLARD, M.A.
Hony. Treasurer.

I have examined the Treasurer's books and vouchers and the monthly statements submitted by the Secretary, Assistant Secretary and Assistant Librarian, and declare the above accounts to be correct.

Madras, 7th February 1914.

(Sd.) K. AMRITA ROW, M.A.,
Auditor.

Budget for 1914.

Receipts.	Rs.	A.	P.	*Expenditure.*	Rs.	A.	P.
To Balance brought forward, less £12/19/9 due to Book-sellers.	776	2	6	By Books and Journals	800	0	0
,, Subscriptions current	1,300	0	0	,, Library charges...	350	0	0
,, do. arrears.	500	0	0	,, Journal Printing.	550	0	0
,, do. Journal.	150	0	0	,, Ordinary working expenses... ...	250	0	0
,, Miscellaneous ...	3	13	6	,, Investment for Capital	500	0	0
				,, Balance to 1915 ...	280	0	0
TOTAL Rs....	2,730	0	0	TOTAL Rs....	2,730	0	0

Madras, 7th February 1914.

(Sd.) C. POLLARD, M.A.,
Hon. Treasurer.

On the Bicircular Quartic.

By A. C. L. Wilkinson, M.A., F.R.A.S.

1. A fundamental theorem of the general bicircular quartic, given by Casey, is the following :—

A bicircular quartic can be generated in four and only four ways as the envelope of a circle which cuts a fixed circle orthogonally and whose centre moves on a fixed central conic. This fixed conic is called a focal conic and the four focal conics corresponding to the four methods of generating the quartic are confocal, their foci being the double foci of the quartic.

The fixed circle will also be called a *circle of inversion*, since the bicircular quartic inverts into itself with respect to any one of these four circles.

Basset's *Elementary Treatise on Cubic and Quartic Curves*, Chap. IX gives an account of these curves. I shall refer to this work by the name of the author simply.

When the fixed conic is a parabola we obtain the Circular Cubics, and when the fixed conic is a circle we obtain the Cartesians, including the so-called Oval of Descartes.

2. Following Salmon's *Higher Plane Curves*, Chapter VI, I consider first the general binodal quartic.

Take for triangle of reference a triangle having the two nodes for two of its vertices; let $z=0$ be the side joining these vertices. Assume that the other vertex of the triangle does not lie on the quartic. The general equation to the binodal is :

$$x^2y^2+2xyz(lx+my)+z^2(ax^2+by^2+cz^2+2hxy+2gxz+2fyz)=0,$$

where c is not zero.

Identify the left hand side with

$$\begin{aligned}(H_1xy+G_1xz+F_1yz)(H_2xy+G_2xz+F_2zy)\\ -C(z^2+Hxy+Gxz+Fyz)^2=0. \qquad \ldots (1)\end{aligned}$$

The following equations arise for determining the ten constants :

$$\begin{aligned}&C=-c,\ F_1G_2+F_2G_1-2CH-2CGF=2h\\ &H_1H_2-CH^2=1,\ F_1H_2+F_2H_1-2CHF=2m\\ &G_1G_2-CG^2=a,\ H_1G_2+H_2G_1-2CHG=2l\\ &F_1F_2-CF^2=b,\ 2CG=-2g,\ 2CF=-f.\end{aligned}$$

We have to see that these equations are consistent and sufficient or the determination of H_1, G_1, F_1 etc.

Now C, G, F are at once determined uniquely and we may re-write the other equations, as follows:

$$H_1H_2=1+CH^2,\ F_1G_2+F_2G_1=2h+2CH+2CGF,$$
$$G_1G_2=a+CG^2,\ F_1H_2+F_2H_1=2m+2CHF,$$
$$F_1F_2=b+CF^2,\ H_1G_2+H_2G_1=2l+2CHF,$$

which are precisely the conditions that

$$(1+CH^2)x^2+(a+CG^2)y^2+(b+CF^2)z^2+2(l+CHG)xy$$
$$+2(m+CHF)xz+2(h+CH+CFG)yz=0$$

may break up into the factors $H_1x+G_1y+F_1z$ and $H_2x+G_2y+F_2z$.

Thus since C, G, F are known, we have for H the equation

$$\begin{vmatrix} 1+CH^2, & l+CGH, & m+CHF \\ l+CGH, & a+CG^2, & h+CH+CGF \\ m+CHF, & h+CH+CGF, & b+CF^2 \end{vmatrix}=0.$$

This is a biquadratic for H, and corresponding to each value of H derived from this equation the factors $H_1x+G_1y+F_1z$ and $H_2x+G_2y+F_2z$ are uniquely determined, except for an arbitrary constant multiplier which does not affect the product.

From this the following theorem results:

The binodal quartic may be exprcesed in the form (1) in precisely four different ways and in general these are distinct.

More generally the quartic can be reduced to the form

$$(C_1z^2+H_1xy+G_1xr+F_1yz)\ (C_2z^2+H_2xy+G_2xz+F_2yz)$$
$$-(Cz^2+Hxy+Gxz+Fyz)^2=0,$$

in an infinite number of ways and it is quite easy to reduce all these to the fundamental form (1).

For, writing this $UV=W^2$, where $U=C_1z^2+++$, $V=C_2z^2+++$ $W=Cz^2+++$, we have

$$(W+\lambda U)^2=U(V+2\lambda W+\lambda^2U),$$
$$\{W+\lambda U+\mu(V+2\lambda W+\lambda^2U)\}^2$$
$$=(V+2\lambda W+\lambda^2U)\{U+2\mu(W+\lambda U)+\mu^2(V+2\lambda W+\lambda^2U)\}.$$

Choosing λ, μ so that the co-fficients of z^2 vanish in both the factors on the right hand side,

$$C_1\lambda^2+2C\lambda+C_2=0,$$
$$C_1+2\mu(C+\lambda C_1)=0.$$

We thus reduce $W^2=UV$; to the findamental form (1).

We cannot have C, C_1 or C, C_2 both zero; nor $C_1C_2=C_1^2$ since the vertex (1,0,0) does not lie on the curve; hence λ, μ can always be found to effect the reduction.

Consider the form $W^2=UV$, this is the envelope of the system of conics

$$\lambda^2U+2\lambda W+V=0. \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The pole of $z=0$ with respect to (2) is given by

$$\lambda^2(H_1y+G_1z)+2\lambda(Hy+Gz)+H_2y+G_2z=0,$$
$$\lambda^2(H_1x+F_1z)+2\lambda(Hx+Fz)+H_2x+F_2z=0,$$

these being the polars of the points (1, 0, 0) and (0, 1,0) respectively. Writing these

$$\frac{x}{\lambda^2F_1+2\lambda F+F_2}=\frac{y}{\lambda^2G_1+2\lambda G+G_2}=\frac{z}{\lambda^2H_1+2\lambda H+H_2}$$

the elimination of λ must give a conic as the locus of this pole. For, any straight line $lx+my+nz=0$ meets the locus in the two points given by the values of λ satisfying the equation

$$l(\lambda^2F_1+2\lambda F+F_2)+m(\lambda^2G_1+2\lambda G+G_2)-n(\lambda^2H_1+2\lambda H+H_2)=0.$$

We may thus state the following general theorem.

The general binodal quartic is the envelope of a conic belonging to the net U, V, W of three conics through two fixed points (and not all passing through a third point) such that the pole of the line joining these two points with respect to the variable conic moves on a fixed conic.

There are four such nets of conics whereby the binodal quartic may be generated.

If the three conics all pass through a third point the corresponding quartic is trinodal having nodes at the three fixed points of intersection.

There are only four such nets; for, we have seen that $W^2=UV$ is identical with

$$\{W+\lambda U+\mu(V+2\lambda W+\lambda^2U)\}^2$$
$$=(V+2\lambda W+\mu^2U)\{U+2\mu(W+\lambda U)+\mu^2(V+2\lambda W+\lambda^2U)\}$$

and the net of conics U, V, W is precisely identical with the net

$$k_1U+k_2V+k_3W,\ k_1'U+k_2'V+k_3'W,\ k_1''U+k_2''V+k_3''W.$$

3. *The four conics, loci of the poles of the variable conics of the nets are inscribed in the quadrilateral formed by the four tangents to the binodal at its nodes.*

Take the fixed conic, locus of the poles, as

$$ax^2+by^2+cz^2+2hxy+2gxz+2fyz=0, \quad \ldots \quad \ldots \quad (3)$$

and the quartic to be the envelope of

$$\lambda(Lyz+Mzx+Nxy+Pz^2)+\mu(L'yz+M'zx+N'xy+P'z^2)$$
$$+\nu(L''yz+M''zx+N''xy+P''z^2)=0,$$

where the pole of this conic with respect to $z=0$ lies on the fixed conic

$$(a, b, c, f, g, h)(x, y\ z)^2=0.$$

The pole of $z=0$ is given by

$$\lambda(Mz+Ny)+\mu(M'z+N'y)+\nu(M''z+N''y)=0,$$
$$\lambda(Lz+Nx)+\mu(L'z+N'x)+\nu(L''z+N''x)=0,$$

this being the intersection of the polars of (0,1,0) and (1,0,0) :

Or $x:y:z=\lambda L+\mu L'+\nu L''\quad \lambda M+\mu M'+\nu M'' :-(\lambda N+\mu N'+\nu N'')$.

Thus we require the envelope of

$$(\lambda L+\mu L'+\nu L'')yz+(\lambda M+\mu M'+\nu M'')zx+(\lambda N+\mu N'+\nu N'')xy$$
$$+(\lambda P+\mu P'+\nu P'')z^2=0,$$

subject to $\qquad (a,\ b.\ c,\ f,\ g,\ h)(L^x,\ M^x,-N^x)^2=0,$

writing $\lambda L+\mu L'+\nu L''=L^x, \lambda M+\mu M'+\nu M''=M^x, \lambda N+\mu N'+\nu N''=N^x$.

In other words, we require the envelope of

$$L^x yz+M^x zx+N^x xy=z^2\begin{vmatrix} 0, & P, & P', & P'' \\ L^x, & L, & L', & L'' \\ M^x, & M, & M', & M'' \\ N^x, & N, & N', & N''\end{vmatrix}\div\begin{vmatrix} L, & L', & L'' \\ M, & M', & M'' \\ N, & N', & N''\end{vmatrix}$$

$$=(\alpha L^x+\beta M^x+\gamma N^x)z^2$$

subject to $\qquad (a,\ b,\ c,\ f,\ g,\ h)\ (L^x,\ M^x,-N^x)^2=0.$

This envelope is

$$(A,\ B,\ C,\ F,\ G,\ H)\ (yz-\alpha z^2,\ zx-\beta z^2,-xy+\gamma z^2)=0,$$

where A, B, C, F, G, H are the usual minors in the discriminant of

$$(a,\ b,\ c,\ f,\ g,\ h)\ (xyz)^2=0.$$

Thus the quartic is

$$A(yz-\alpha z^2)^2+B(zx-\beta z^2)^2+C(xy-\gamma z^2)^2+2H(yz-\alpha z^2)\ (zx-\beta z^2)$$
$$-2F(zx-\beta z^2)(xy-\gamma z^2)-2G(yz-\alpha z^2)(xy-\gamma z^2)=0.$$

The tangents at the nodes $y=0,\ z=0$ and $x=0,\ z=0$ are given by

$$Bz^2+Cy^2-2Fyz=0,$$
$$Az^2+Cx^2-2Gxz=0.$$

Now the condition $\lambda x+\mu y+\nu z=0$ may touch the conic (3) is

$$(A,\ B,\ C,\ F,\ G,\ H)\ (\lambda,\ \mu,\ \nu)^2=0.$$

If therefore $\lambda x+\nu z=0$ is a tangent from $z=0,\ x=0$, we get for the pair of tangents from (0, 1, 0) the equations

$$(A,\ B,\ C,\ F,\ G,\ H)\ (z,\ o,-x)^2=0,$$

or

$$Az^2+Cx^2-2Gxz=0.$$

Similarly the tangents from (1,0,0) are given by

$$Bz^2+Cy^2-2Fyz=0.$$

Thus the conic, locus of the poles, is inscribed in the quadrilateral formed by the four tangents at the nodes and thus the four directing

conics are all inscribed in the same quadrilateral. When the nodes are at the circular points at infinity, the conics U, V, W of § 2 are circles; and a net of circles, say $\lambda C_1+\mu C_2+\nu C_3=0$, where C_1, C_2, C_3 are fixed circles, can be expressed otherwise. There is a single circle C that cuts orthogonally C_1, C_2, C_3 and also every circle of the form $\lambda C_1+\mu C_2+\nu C_3=0$, and conversely. Hence we have the theorem stated in § 1:

A bicircular quartic can in general be generated in four different ways as the envelope of a circle that cuts a fixed circle orthogonally and whose centre moves on a fixed conic. The four fixed conics so obtained are confocal, their foci being the double foci of the bicircular quartic.

4. Consider a circular cubic and the line at infinity as the limit of a bicircular quartic. We obtain the theorem:

A circular cubic can be generated as the envelope of a circle which cuts a fixed circle orthogonally and whose centre moves on a fixed parabola.

This can be done in general in four ways and the parabolas are confocal.

Again, the trinodal quartic may be looked upon as the envelope of $\lambda yz+\mu zx+\nu xy=0$, subject to the condition

$$a\lambda^2+b^2\mu+c\nu^2+2h\lambda\mu+2g\lambda\nu+2f\mu\nu=0.$$

The trinodal is

$$Ay^2z^2+Bz^2x^2+Cx^2y^2+2Hxyz^2+2Gxy^2z+2Fx^2yz=0.$$

The six tangents at the nodes are

$$Ay^2+Bx^2+2Hxy=0,$$
$$Az^2+Cx^2+3Gxz=0,$$
$$Bz^2+Cy^2+2Fyz=0.$$

As in § (3) these six tangents touch the conic whose tangential equation is $(A, B, C, -F, -G, -H)(\lambda, \mu, \nu)^2=0$, and the point equation is therefore

$$(BC-F^2)x^2+\ldots 2(FG+CH)xy+\ldots=0.$$

This proof is the same virtually as that in Basset, § 192.

5. The general theorems obtained give no indication respecting the reality of the focal conic or fixed circles. We proceed to determine the reality of the four methods of generation and the corresponding classification of bicircular quartics.

The general real bicircular quartic is given by

$$(x^2+y^2)^2+4(x^2+y^2)(lx+my)+ax^2+2hxy+by^2+2gx+2fy+c=0,$$

which can be written

$$(x^2+y^2+2lx+2my+l^2+m^2)^2=a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'.$$

Changing the origin, this becomes

$$(x^2+y^2)^2=ax^2+2hxy+by^2+2gx+2fy+c, \text{ say.}$$

If the discriminant of the right hand side is not zero, consider the equation

$$(x^2+y^2+\delta)^2=(a+2\delta)x^2+2hxy+(b+2\delta)y^2+2gx+2fy+c+\delta^2.$$

The discriminant of the right hand side will vanish, if

$$(a+2\delta)(b+2\delta)(c+\delta^2)+2fgh-(a+2\delta)f^2-(b+2\delta)g^2-(c+\delta^2)h^2=0.$$

The roots of the equation $(a+2\delta)(b+2\delta)-h^2=0$ are real and they make the remaining terms $(a+2\delta)f^2+(b+2\delta)g^2-2fgh$ a perfect square. Just as in the discussion of the discriminating cubic in Solid Geometry, these squares are one positive and one negative. Therefore the equation in δ has at least one real root. Thus by transformation of axes we can reduce the bicircular quartic to the standard form

$$(x^2+y^2+2gx+2fy+c)^2=ax^2+by^2\ ;$$

where, in general, neither a nor b is zero. If one of a, b is zero, the bicircular degenerates to a pair of circles.

If, however, $h=0$ and $a=b$, the equation

$$(x^2+y^2+\delta)^2=(a+2\delta)(x^2+y^2)+2gx+2fy+c+\delta^2$$

which represents a cartesian, will have its right hand side a point circle if $(a+2\delta)\ \{\ (a+2\delta)(c+\delta^2)-f^2-g^2\ \}=0$, and the cubic in δ has then one real root.

Thus every real bicircular quartic, including cartesians, can be reduced to the form

$$(x^2+y^2-c)^2=a(x-\alpha)^2+b(y-\beta)^2.$$

6. Consider now the bicircular quartic generated by a circle cutting $(x-\alpha)^2+(y-\beta)^2=\gamma^2$ orthogonally and whose centre describes the conic $\frac{x^2}{a}+\frac{y^2}{b}=1$. Write $a_1=\sqrt{a}$, $b_1=\sqrt{b}$.

The circle $(x-a_1\cos\ \theta)^2+(y-b_1\sin\ \theta)=r^2$ cuts the given circle orthogonally, if

$$(\alpha-a_1\cos\theta)^2+(\beta-b_1\sin\theta)^2=\gamma^2+r^2.$$

Subtracting, we require therefore to find the envelope of the system of circles

$$x^2+y^2-2(x-\alpha)a_1\cos\theta-2(y-\beta)b_1\sin\theta-\alpha^2-\beta^2+\gamma^2=0 \quad \ldots (1)$$

The envelope is obtained by eliminating θ between this equation and

$$(x-\alpha)a_1\sin\theta-(y-\beta)b_1\cos\theta=0. \quad \ldots (2)$$

The envelope is therefore

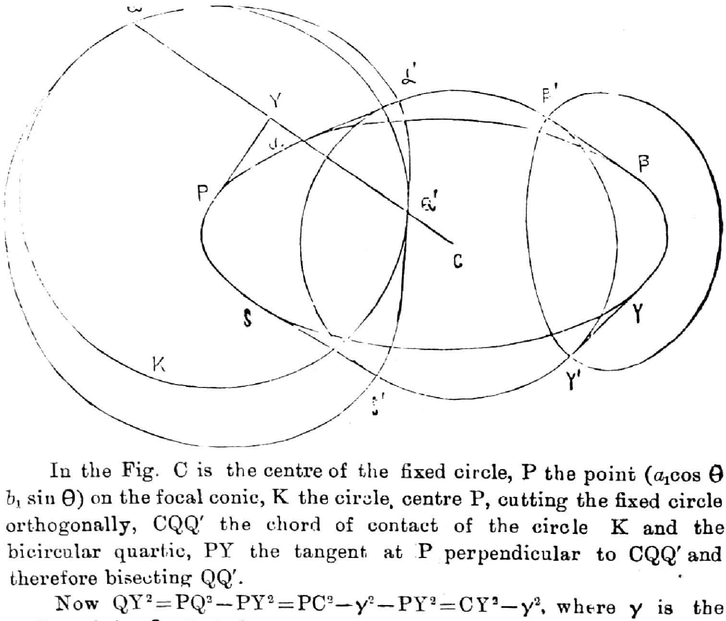
$$(x^2+y^2-\alpha^2-\beta^2+\gamma^2)^2=4a(x-\alpha)^2+4b(y-\beta)^2. \quad \ldots (3)$$

Hence we derive the theorem :

The bicircular quartic $(x^2+y^2-c)^2=4a(x-\alpha)^2+4b(y-\beta)^2$ *has* $\frac{x^2}{a}+\frac{y^2}{b}=1$ *for a focal conic, and* $(x-\alpha)^2+(y-\beta)^2=\alpha^2+\beta^2-c$, *for the corresponding fixed circle.*

The circle is real only if $\alpha^2+\beta^2>c$.

Some interesting geometrical theorems and another method of generating the bicircular can be now obtained. The chord of contact of the generating circle (1) with the bicircular (3) is the straight line (2) which passes through the centre (α, β) of the fixed circle and is perpendicular to $\frac{x}{a_1}\cos\theta+\frac{y}{b_1}\sin\theta=1$ which is the tangent at the point $(a_1\cos\theta, b_1\sin\theta)$ to the focal ellipse.



In the Fig. C is the centre of the fixed circle, P the point $(a_1\cos\theta$ $b_1\sin\theta)$ on the focal conic, K the circle, centre P, cutting the fixed circle orthogonally, CQQ′ the chord of contact of the circle K and the bicircular quartic, PY the tangent at P perpendicular to CQQ′ and therefore bisecting QQ′.

Now $QY^2=PQ^2-PY^2=PC^2-\gamma^2-PY^2=CY^2-\gamma^2$, where γ is the radius of the fixed circle.

We have therefore a second method of generating a general bicircular quartic :

From a fixed point C draw the perpendicular CY on the tangent to a conic, the locus of points Q on CY such that CY^2-QY^2 is constant, is a bicircular quartic.

If CY^2-QY^2 is positive the corresponding fixed circle is real, but if CY^2-QY^2 is negative the corresponding fixed circle is imaginary.

The fixed circle we have hitherto spoken of is also called the *circle of inversion* ; for, if the bicircular quartic is inverted with respect to it

the inverse figure is the same bicircular quartic. This is obvious since $CQ \cdot CQ' = \gamma^2$.

There are four points with respect to which a bicircular quartic can be inverted into itself.

The normals, at Q, Q′ to the bicircular intersect on the focal conic at P. Basset, §206.

7. Consider the bicircular

$$(x^2+y^2-c)^2 = 4a(x-\alpha)^2 + 4b(y-\beta)^2$$

Identifying this with

$$(x^2+y^2-c') = 4(a+\lambda)(x-\alpha')^2 + 4(b+\lambda)(y-\beta')^2$$

we have

$$2c'+4\lambda = 2c$$

$$c'^2 - 4a\alpha'^2 - 4b\beta'^2 - 4\lambda(\alpha'^2+\beta'^2) = c^2 - 4a\alpha^2 - 4b\beta^2$$

$$a\alpha' + \lambda\alpha' = a\alpha$$

$$b\beta' + \lambda\beta' = b\beta.$$

Hence

$$2(c'-c) = -4\lambda$$

$$a(\alpha'-\alpha) = -\lambda\alpha'$$

$$b(\beta'-\beta) = -\lambda\beta'$$

$$c'^2 - c^2 - 4a(\alpha'^2-\alpha^2) - 4b(\beta'^2-\beta^2) - 4\lambda(\alpha'^2+\beta'^2) = 0.$$

By use of the first three equations, the last becomes

$$c'+c-2\alpha'(\alpha+\alpha') - 2\beta'(\beta+\beta') + 2\alpha'^2 + 2\beta'^2 = 0,$$

or finally

$$c = \lambda + \frac{a\alpha^2}{a+\lambda} + \frac{b\beta^2}{b+\lambda} \quad \ldots \quad \ldots \quad (1)$$

$$\alpha' = \frac{a\alpha}{a+\lambda}, \quad \beta' = \frac{b\beta}{b+\lambda}, \quad c' = c - 2\lambda.$$

The cubic in λ gives the three other methods of generating the bicircular by means of the focal conics $\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} = 1$, and the circles $\left(x - \frac{a\alpha}{a+\lambda}\right)^2 + \left(y - \frac{b\beta}{b+\lambda}\right)^2 = \frac{a^2\alpha^2}{(a+\lambda)^2} + \frac{b^2\beta^2}{(b+\lambda)^2} - c + 2\lambda$, where λ is a root of (1).

These fixed circles are mutually orthogonal.

For, consider the two circles

$$x^2+y^2-2\alpha x - 2\beta y + c = 0,$$

$$x^2+y^2 - \frac{2a\alpha}{a+\lambda}x - \frac{2b\beta}{b+\lambda}y + c - 2\lambda = 0;$$

these are orthogonal, if

$\frac{a\alpha^2}{a+\lambda} + \frac{b\beta^2}{b+\lambda} - c + \lambda = 0$, which is condition (1).

8. The bitangents of the bicircular

$$(x^2+y^2-c)^2=a(x-\alpha)^2+b(y-\beta)^2$$

are given by
$$a(x-\alpha)^2+b(y-\beta)^2=0,$$
and
$$(a+\lambda)(x-\alpha')^2+(b+\lambda)(y-\beta')^2=0,$$
where λ, α', β' are determined as in § 5.

The points of contact lie on the four concentric circles

$$x^2+y^2-c=0,\ x^2+y^2-c+2\lambda=0,$$

where λ is a root of (1). Also

$$2\{a(x-\alpha)^2+b(y-\beta)^2\}+\lambda(x^2+y^2-c),$$
$$\equiv 2\{a+\lambda)(x-\alpha')^2+(b+\lambda)(y-\beta')^2\}-\lambda(x^2+y^2-c+2\lambda)\}$$

in virtue of the relations of § 5.

Thus the eight points of contact of the two pairs of bitangents lie on the conic

$$2\{a(x-\alpha)^2+b(y+\beta)^2\}+\lambda(x^2+y^2-c)=0.$$

A similar theorem is true for the general quartic.

If λ_1, λ_2, λ_3 are the roots of the equation,

$$\frac{a\alpha^2}{a+\lambda}+\frac{b\beta^2}{b+\lambda}=c-\lambda,$$

we obtain

$$\frac{a\alpha^2}{(a+\lambda_1)(a+\lambda_2)}+\frac{b\beta^2}{(b+\lambda_2)(b+\lambda_2)}=\frac{1}{4}$$

$$\frac{a\alpha^2}{(a+\lambda_1)(a+\lambda_2)(a+\lambda_3)}+\frac{b\beta^2}{(b+\lambda_1)(b+\lambda^2)(b+\lambda_3)}=0, \qquad \dots (1)$$

Every real bicircular quartic has two real bitangents.

For, if a, b have opposite signs the pair of straight lines $a(x-\alpha)^2+b(y-\beta)^2=0$ is real.

If a, b have the same sign then, if $\lambda_1\lambda_2\lambda_3$ are all real, from (1) at least one pair of values $a+\lambda$, $b+\lambda$ must have opposite signs; if, however, λ_1, λ_2, are conjugate imagineries, then both $(a+\lambda_1)(a+\lambda_2)$ and $(b+\lambda_1)(b+\lambda_2)$ are positive and thus from (1) $a+\lambda_3$, $b+\lambda_{32}$ have opposite signs.

A real bicircular quartic has either two real or six real bitangents, but cannot have four or eight.

For, from (1) when $\lambda_1,\lambda_2,\lambda_3$ are all real the signs of the pairs a, b; $a+\lambda_1$, $b+\lambda_1$; $a+\lambda_2$, $b+\lambda_2$, $a+\lambda_3$, $b+\lambda_3$ must be either three alike and one different, or one alike and three different; when two of the values of λ are imaginary, say λ_1, λ_2, the pairs a, b; $(a+\lambda_3)$, $(b+\lambda_3)$; must be one alike and one different in signs.

9. I proceed now to discuss the generation of the bicircular quartic by means of real focal conics and real circles of inversion.

Since there is always one pair of imaginary and therefore conjugate imaginary bitangents, every real bicircular quartic can be written

$$(x^2+y^2-c)^2=4a^2(x-\alpha)^2+4b^2(y-\beta)^2$$

where all the letters denote real quantities.

Corresponding to the equivalent form

$$(x^2+y^2-c+2\lambda)^2=4(a^2+\lambda)(x-\alpha')^2+4(b^2+\lambda)(y-\beta')^2$$

we have

$$\alpha'=\frac{a^2\alpha}{a^2+\lambda},\ \beta'=\frac{b^2\beta}{b^2+\lambda},\ c=\lambda+\frac{a^2\alpha^2}{a^2+\lambda}+\frac{b^2\beta^2}{b^2+\lambda},$$

and the radius of the corresponding circle of inversion is

$$\gamma'^2=\alpha'^2+\beta'^2-c+2\lambda$$

$$=\frac{a^4\alpha^2}{(a^2+\lambda)^2}+\frac{b^4\beta^2}{(b^2+\lambda)^2}-\frac{a^2\alpha^2}{a^2+\lambda}-\frac{b^2\beta^2}{b^2+\lambda}+\lambda$$

$$=\lambda\left\{1-\frac{a^2\alpha^2}{(a^2+\lambda)^2}-\frac{b^2\beta^2}{(b^2+\lambda)^2}\right\}=\lambda\frac{\partial c}{\partial\lambda}.$$

Consider now the curve

$$y=x+\frac{a^2\alpha^2}{a^2+x}+\frac{b^2\beta^2}{b^2+x}.$$

It has asymptotes $y=z$, $x+a^2=0$, $x+b^2=0$.

The turning points are given by the equation

$$1-\frac{a^2\alpha^2}{(a^2+x)^2}-\frac{b^2\beta^2}{(b^2+x)^2}=0,$$

or

$$(a^2+x)^2(b^2+x)^2-a^2\alpha^2(b^2+x)^2-b^2\beta^2(a^2+x)^2=0.$$

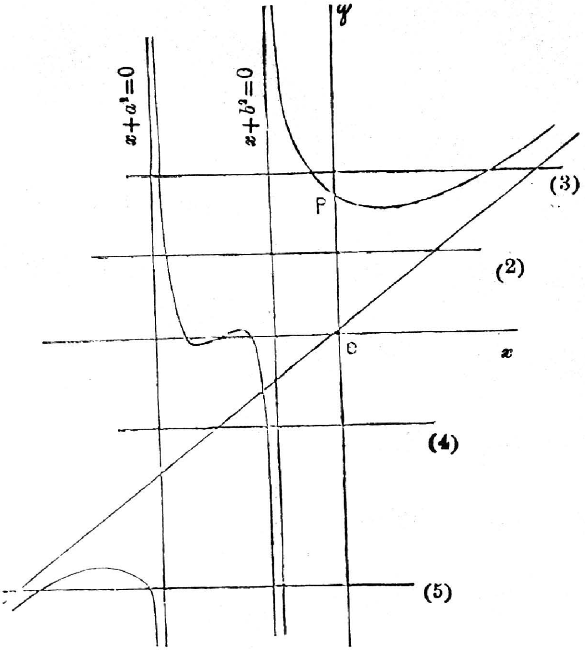
The signs of the function on the left hand side are

for	$x=+\infty$,	$x=-b^2$,	$x=-a^2$,	$x=-\infty$.
	$+$,	$-$,	$-$,	$+$.

There is therefore one turning point on each of the branches having $y=x$ for asymptote and from geometrical considerations there cannot be three turning points on either of these branches.

Thus the most general case occurs when this equation has four real roots. The corresponding curve is represented in the Fig.

When $x=0$, $OP=\alpha^2+\beta^2$.



Consider the intersections of the curve with a variable straight line $y=c$, then at the points of intersection the following cases arise.

I. A straight line close to the axis of x intersects the branch of the curve lying between the parallel asymptotes in three real points.

Let the values of λ from left to right be λ_1, λ_2, λ^3.

II. A straight line as (2) or (4) in the figure intersects the curve in only one real point.

III. A straight line below P with three real intersections λ_1, λ_2, λ^3 from left to right.

(1) The turning point on the branch is to the right of P.

(2) The turning point on the branch is to the left of P.

IV. A straight line above P as (3) in the figure with three real intersections λ_1, λ_2, λ_3 from left to right.

V. A straight line as (5). At the first two points of intersection from the left, both $a^2+\lambda$ and $b^2+\lambda$ are negative. The bicircular is therefore purely imaginary.

The following table gives the signs of $a^2+\lambda$, $b^2+\lambda$, $y'^2=\lambda\frac{\partial c}{\partial \lambda}=x\frac{dy}{dx}$ for the point of intersection, $\in$, and $y^2=\alpha^2+\beta^2-c$.

Case.	$a^2+\lambda$	$b^2+\lambda$	y'^2	c	y^2
I λ_1	+	−	+	$\pm<\alpha^2+\beta^2$	+
,, λ_2	+	−	−	,,	+
,, λ_3	+	−	+	,,	+
II	+	−	+	$\pm<\alpha^2+\beta^2$	+
III (1) λ_1	+	−	+	$+<\alpha^2+\beta^2$	+
,, λ_2	+	+	−	,,	+
,, λ_3	+	+	+	,,	+
III (2) λ_1	+	−	+	,,	+
,, λ_2	+	+	+	,,	+
,, λ_3	+	+	−	,,	+
IV λ_1	+	−	+	$+>\alpha^2+\beta^2$	−
,, λ_4	+	+	+	,,	−
,, λ_3	+	+	+	,,	−

An inspection of the table leads to the following conclusions:

1. *When the equation in λ has only one real root, the bicircular quartic can be generated in two ways only. One by a real focal ellipse and real circle of inversion, the other by a real focal hyperbola and real circle of inversion. There are only two real bitangents.*

2. *When the equation in λ has three real roots, all the focal conics are real, but only three of the circles of inversion are real.* (1) *The bicircular*

can be generated from two real focal ellipses and corresponding real circles of inversion, or from one real focal hyperbola and corresponding real circle of inversion. There are only two real bitangents. (2) *The bicircular can be generated from one real focal ellipse and corresponding real circle of inversion or from two real focal hyperbolas and corresponding real circle of inversion. There are six real bitangents.*

Considering the second method of generation in § 6, where $CY^2-QY^2=y^2$: we see that in (1) the bicircular can be generated from either an ellipse or a hyperbola; in (2) the bicircular can be generated in four real ways one of which is always from an ellipse and another always from a hyperbola.

10. It may be of interest to consider a numerical example and we select a case of type IV.

Consider the bicircular given by

$$\left(x^2+y^2-\frac{5}{2}\right)^2=12\left(x-\frac{2}{\sqrt{3}}\right)^2+8\left(y-\frac{\sqrt{3}}{2}\right)^2.$$

The equation in λ is

$$\frac{5}{2}=\lambda+\frac{4}{3+\lambda}+\frac{3}{4+2\lambda}.$$

This has roots $\pm 1, -\frac{5}{2}$.

For $\lambda=1$, $\alpha'=\frac{\sqrt{3}}{2}$, $\beta'=\sqrt{3}$, $c'=\frac{1}{2}$,

" $\lambda=-1$, $\alpha'=\sqrt{3}$, $\beta'=\sqrt{3}$, $c'=\frac{9}{2}$,

" $\lambda=-\frac{5}{2}$, $\alpha'=4\sqrt{3}$, $\beta'=-2\sqrt{3}$, $c'=\frac{15}{2}$

The equivalent forms of the bicircular quartic are, therefore,

$$\left(x^2+y^2-\frac{1}{2}\right)^2=16\left(x-\frac{\sqrt{3}}{2}\right)^2+12\left(y-\frac{1}{\sqrt{3}}\right)^2$$

$$\left(x^2+y^2-\frac{9}{2}\right)^2=8(x-\sqrt{3})^2+4(y-\sqrt{3})^2$$

$$\left(x^2+y^2-\frac{15}{2}\right)^2=2(x-4\sqrt{3})^2-2(y+2\sqrt{3})^2.$$

All these reduce to

$$(x^2+y^2)^2=17\,x^2+13\,y^2-16\sqrt{3}\,x-8\,\sqrt{3}\,y+15\tfrac{3}{4}.$$

The corresponding methods of generation are by means of the respective focal conics and fixed circles of inversion given by :

$$\frac{x^2}{3}+\frac{y^2}{2}=1\,;\; x^2+y^2-\frac{4}{\sqrt{3}}x-\sqrt{3}\,y+\frac{5}{2}=0,$$

$$\frac{x^2}{4}+\frac{y^2}{3}=1\,;\; x^2+y^2-\sqrt{3}\,x-\frac{2}{\sqrt{3}}y+\frac{1}{2}=0,$$

$$\frac{x^2}{2}+\frac{y^2}{1}=1\,;\; x^2+y^2-2\sqrt{3}\,x-2\sqrt{3}\,y+\frac{9}{2}=0,$$

$$2x^2-2y^2=1\,;\; x^2+y^2-8\sqrt{3}\,x+4\sqrt{3}\,y-\frac{15}{2}=0.$$

The first circle only is imaginary ; the only real bitangents are

$$x-4\sqrt{3}=\pm(y+2\sqrt{3}).$$

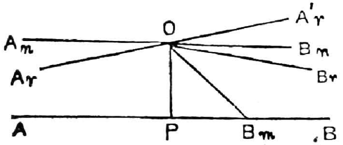
(*To be continued.*)

SHORT NOTES.

Parallel Straight Lines.

[The following is the abstract of a paper read by the writer before the Mathematical Association of the Presidency College, Madras, in January last. Reference: Coolidge's *Non-Euclidean Geometry*.]

Let O be a point and AB a line. Draw OP perpendicular to AB.



Consider the rays OB_n lying in one of the *half*-planes bounded by OP. Then the following alternatives are possible:

(A) *None of the rays OB_n meets PB,*

(R) *All the rays OB_n meet PB,*

(EL) *Some of the rays OB_n meet PB, and some do not meet it.*

Case (A) can be discarded. For, it is tantamount to saying that we cannot connect O and a point in PB by a line.

Case (R) applies to finite space, as can be easily seen. Draw OA_n, OB_n on different sides of OP, making equal angles with it. Then by congruence we prove that OA_n, OB_n meet AB at equal distance from P. Take $P\hat{O}A_n = P\hat{O}B = 90°$ and we deduce that the perpendiculars to the same line OP meet at equal distances on opposite sides of it. Calling this distance $\triangle$, we easily prove that $\triangle$ is independent of the length of OP. If we further agree that two lines cannot have more than one point in common, then the two points of intersection above are identical, and

"*All straight lines are of constant length* $2\triangle$."

This is the assumption of *Riemann* and his geometry is therefore one of finite space.

Case (EL). Let OB_m be any ray which meets PB, and let OB_n be any ray which does not meet it. Then, evidently, OB_n cannot lie in the angle POB_m. Therefore

$$P\hat{O}B_n > POB_m,$$

and, by Dedekind's principle, we can form a cut determining uniquely a ray OB_r, which separates the rays which *meet* PB from the rays which *do not meet* it, and which itself may be assigned to either class,

Def: The ray OB_r thus determined is defined to be *parallel* to the ray PB at the point O, and the angle POB is the *angle of parallelism* for the distance OP.

From symmetry there will be a ray OA_r in the other *half*-plane parallel to the ray PA, so that

$$P\hat{O}A_r = P\hat{O}B_r = \text{angle of parallelism.}$$

Now, either

(E) *The rays OA_r, OB_r are the same line,*

or (L) *They are different lines.*

(1) The former case gives one and only one line parallel to a given line through a given point, and is the ordinary *Euclidean* assumption. The angle of parallelism here is a right angle.

(2) The latter gives two lines through any point parallel to a given line, and is the *Lobatchewskian* assumption. The angle of parallelism in this case is less than a right angle.

If A_rO is produced to A_r', then rays which lie within the angle. B_rOA_r' lie also within the regetically opposite angle, and therefore do not meet AB at either end.

Def: Such lines, as the above, which neither intersect nor are parallel are defined to be *skew lines*.

The following theorems relate to (L) and (E):

THEOREM I. *Parallel rays meet at infinity.*

This is at once obvious. For, the parallel ray through O belongs both to the set which meets PB and to the set which does not meet it (at a finite distance).

Cor. 1. In (E) the rays OA_r, OB_r, being the same can have only one point common with AB. Thus a line has only one point at infinity.

In (L) the rays OA_r, OB_r are different and therefore the rays PA, PB have distinct points at infinity.

[In (R) the points at infinity are imaginary.]

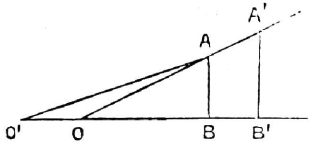
Cor. 2. If a ray be parallel to another ray at one point in it, it is so also at any other point in it. If a ray is parallel to another, the second is parallel to the first. Rays parallel to the same ray are parallel to one another.

These results follow by considering the properties of the point at infinity.

THEOREM II. (*a*) *The distance between two intersecting rays increases as the rays are produced and becomes ultimately infinite*; (*b*) *the distance between two parallel rays decreases in the direction of parallelism, and becomes ultimately zero*; and (*c*) *the distance between two skew lines becomes ultimately infinite both ways.*

(*a*) Let OA, OB be two intersecting rays. From points A, A′ on OA drop perpendiculars AB, A′B′ on OB. Let OB′>OB. Then we have to prove that A′B′>AB.

If not, suppose A′B′=AB. Apply OA′B′ to OAB so that A′B′ coincides with AB and B′O falls along BO. Then O falls without BO, on O′ say. And thus



$$A\hat{O}B > A\hat{O'}B \text{ or } A'\hat{O}B',$$

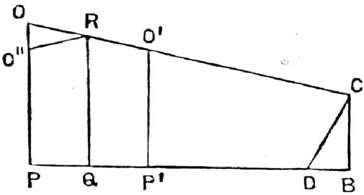
which is absurd.

If A′B′<AB, produce B′A′ to A″, so that A″B′=AB, then by the same procedure we reach a similar absurdity.

Hence A′B′>AB.

To prove that there is no superior limit to A′B′: suppose *l* to be such a limit. Choose OB′ so great that *l* is negligible in comparison with OB′. Then A′OB′ is negligible in comparison with OB′A′ or a right angle, which need not be true. Hence, there cannot be a limit *l*.

(*b*) Let OC, PB be two parallel rays, OP being ⊥ to PB. From a point O′ in OC draw O′P′ ⊥ to PB. Let QR be the ⊥ʳ bisector of PP′. Then since QRO′ is an angle of parallelism in (L), it is less than a right angle.



$$\therefore\ Q\hat{R}O' < Q\hat{R}O.$$

By folding the quad. OPP′O′ along QR, it is obvious that O′ falls within OP.

$$\therefore\ O'P' < OP.$$

To shew that there is no inferior limit to O′P′; suppose *l* to be such a limit. Take PD sufficiently large in PB and draw DC ⊥ to OC,

CB⊥ to PB. Then by Th. I. OD can be made to coincide with OC in the limit. Choose PD so large that $DC < l$.

$$\therefore \qquad CB < CD < l.$$

In other words, there is no inferior limit to the approach of OC and PB.

(*c*) Let AB, CD be two skew lines. Draw AK parallel to CD. Then by (*a*) the distance between AB, AK is ultimately infinite, and by (*b*) that between AK and CD is ultimately zero. Also, since AB is skew to CD, AK lies between AB and CD. Hence the distance between AB, CD is ultimately infinite towards B, D.

Similarly, it is infinite towards the opposite side.

Cor. 1. The distance between two parallel straight lines increases infinitely in the direction opposite to that of parallelism.

Cor. 2. A rectangle cannot exist in (L). For, if $OPP'O'$ be a rectangle, by folding across QR, it is seen that O'' must coincide with O, since $RO''P$ is otherwise greater than ROP. Thus, the opposite sides $O'P'$ and OP should be equal.

Also, from the congruence of OPQR and $O'P'QR$, it follows that OPQR is a rectangle. Therefore

$$OP = RQ = O'P'.$$

By repeating a similar construction in the case of the rectangles OPQR, $RQP'O'$, we find the opposite sides OO', PP' of the rectangle $OPP'O'$ preserve a *constant* distance between them.

We have seen that such lines do not exist in (L). Hence the result.

[A rectangle does not exist in (R) for the same reason.]

Theorem III. *Two skew lines have one and only one common perpendicular which is the shortest distance between them.*

For, since two skew lines diverge infinitely both ways the distance between them must be a minimum at least once. Also, it is readily seen that the common perpendicular is a minimum. Hence there must be at least one common perpendicular. There cannot be more than one; for, then a rectangle will exist.

Hence the theorem.

R. Vythynathaswamy.

Note on the Normal Section of an Enveloping Cylinder.

In the January (1914) number of the Edinburgh Mathematical Society's *Mathematical Notes*, Dr. R. J. T. Bell has an interesting paper on the Principal Axes of a Normal Section of an Enveloping Cylinder of

an Ellipsoid. His method of reasoning in the concluding part of the paper, however, does not seem to be quite convincing.

Having proved that

(1) Any radius vector of an ellipse is equal to the perpendicular from the centre to a tangent;

(2) The central perpendiculars to the tangent planes of an enveloping cylinder are equal to the semi-diameters of a certain section of the ellipsoid;

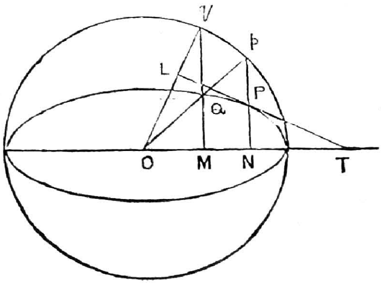
he concludes that the section of the ellipsoid must be equal to the normal section of the cylinder.

Now, for the equality of two ellipses it is not enough that any semi-diameter of one is equal to a semi-diameter of the other; but a further condition, such as the equality of angles between corresponding semi-diameter is necessary. For instance, in the ellipses (a, b) (a',b') any diameter of the first is equal to a diameter of the second, if $a'>a>b>b'$, though the ellipses are distinct. The additional condition required in the present case is easily supplied by the following further property of the ellipse:

The central perpendiculars on conjugate diameters are equal to two perpendicular semi-diameters of the ellipse and have the sum of the squares of their reciprocals constant.

From this we deduce that the sum of the squares of the reciprocals of the principal axes is the same in the two sections considered by Dr. Bell, whence the equality of the sections is established.

The properties referred to above admit of easy geometrical demonstration as under:—



(i) In the figure OL is the central perpendicular on the tangent at a point P of an ellipse, corresponding to p on the auxliary circle and Op

meet the ellipse at Q. To prove that OL meets the auxiliary circle at q which corresponds to Q.

We have

$$\frac{OM}{OQ}=\frac{ON}{OP}=\frac{OP}{OL}=\frac{OQ}{OT}=\frac{OM}{OL}, \text{ since } qLMT \text{ is cyclic.}$$

$$\therefore \qquad OQ=OL.$$

In other words, *the perpendicular on the tangent at P is equal to the semi-diameter of the ellipse in the direction Op.*

(ii) In the ellipsoid, let p (al, am, an) be the point of the auxiliary sphere corresponding to a point P (al, bm, cn) of the ellipsoid. Then if Op meet the ellipsoid at Q (al', bm', cn'), the corresponding point q (al', am', an') of the sphere is on the central perpendicular OL to the tangent plane at P, and OQ$=$OL.

For the proof, the steps are exactly the same as in (i) except that the point P in the figure is, in general, outside the plane of the paper.

(iii) The normal section of the enveloping cylinder whose generators are parallel to OQ is of area A′, such that

$$OQ \cdot A' = \pi\, abc. \text{ (Property of conjugate diameters.)}$$

Also, the central section A conjugate to OP is such that

$$OL.A=\pi\, abc.$$

Hence $$OQ\cdot A'=OL\cdot A.$$

Thus, the sections A and A′ are equal.

1st *March* 1914. M. T. NARANIENGAR.

The Face of the Sky for May and June 1914.

Sidereal time at 8 p.m.

	May.			*June.*	
D.	H.	M.		H.	M.
1	10	36	...	0	38
8	11	3	...	1	5
15	11	31	...	1	33
22	11	58	...	2	1
29	12	26	...	2	28

From this table the constellations visible during the evenings of May and June can be ascertained by a reference to the positions as given in a star-atlas.

The Sun

enters Cancer on June 22 at 0-25 P.M.

Phases of the Moon.

	May.				*June.*		
	D.	H.	M.		D.	H.	M.
First Quarter ...	3	11	59	...	1	7	33
Full Moon ...	10	3	1	...	8	10	48
Last Quarter ...	17	3	42	...	15	7	50
New Moon ...	25	8	5	...	23	9	3

Planets.

Mercury attains its greatest elongation (E) on June 1914. It is in conjunction with the Moon on May 26 at 6-37 A.M., and on June 25 and with Neptune on June 26.

Venus which is an evening star, is in conjunction with the Moon on May 27 and with Neptune on June 17.

Mars is in conjunction with the Moon on May 31 and June 28. It is visible in the evenings.

Jupiter is in quadrature on May 12; is stationary on June 12 when it begins to retrograde. It is in conjunction with the Moon on June 13.

Saturn is in conjunction with the Moon on June 23.

Uranus is in quadrature to the Sun on May 3 and stationary on May 17. It is in conjunction with the Moon on May 16.

Neptune is in conjunction with the Sun on May 29 and with the Moon on May 2.

V. RAMESAM.

SOLUTIONS.

Question 494.

(S. NARAYANA AIYAR, M.A. :—Prove that

$$\int_0^\infty \frac{\phi(a+ix)-\phi(a-ix)}{2\,ix}dx=\frac{\pi}{2}\left\{\phi(a)-\phi(\infty)\right\}.$$

Solution by R. Srinivasan, M.A., and R. Vythynathaswamy.

By Taylor's Theorem—

$$\phi(a+ix)-\phi(a-ix)=\sum\frac{a^n}{|n}\left\{\phi^n(ix)-\phi^n(-ix)\right\}.$$

$$\therefore\qquad 1=\sum\frac{a^n}{|n\cdot 2i}\int_0^\infty \frac{\phi^n(ix)-\phi^n(-ix)}{x}dx$$

$$=\sum\frac{a^n}{|n\cdot 2i}\left\{\phi^n(\infty)-\phi^n(o)\right\}\log\frac{ib^2}{-i},$$

$$=\frac{\log(-1)}{2i}\left\{\phi(a+\infty)-\phi(a+0)\right\},$$

by the theorem of Frullani, as modified by Elliot.

But $\log(-1)=\log e^{-i\pi}=-i\pi$.

Hence $$1=\frac{\pi}{2}\left\{\phi(a)-\phi(\infty)\right\}.$$

Question 497.

(T. P. TRIVEDI, M.A., L.L.B.,) :—Prove that

$$\int_0^\infty \frac{\sin rx}{x(1+x^2)^2}dx=\frac{\pi}{4}\left(2-e^{-r}(r+2)\right)$$

$$\int_0^\infty \frac{\sin^2 rx}{x^2(1+x^2)}dx=\frac{\pi}{4}\left(-1+e^{-2r}+2r.\right)$$

Solution (1) *by K. Narasinga Rao,* (2) *by V. K. Aravamudan and others.*

(1) We know that $\int_0^\infty \frac{\cos rx}{(a^2+x^2)}dx=\frac{\pi}{2a}e^{-ar}$ (Williamson, p. 151).

Differentiating with respect to a, we get

$$\int_0^\infty \frac{\cos rx}{(a^2+x^2)^2}dx=\frac{\pi e^{-ar}}{4a^3}(1+ar)$$

Integrating with respect to r

$$\int_0^\infty \frac{\sin rx}{x(a^2+x^2)^2}dx=\frac{\pi}{4a^3}\left[\int e^{-ar}\,dr+\int are^{-ar}\,dr\right]$$

$$=\frac{\pi}{4a^3}\left[-\frac{e^{-ar}}{a}-\frac{e^{-ar}}{a}-ar\frac{e^{-ar}}{a}\right]+\text{const.}$$

$$=-\frac{\pi}{4a^4}(ar+2)\,e^{-ar}+\text{const.}$$

Put $r=0$, and the constant is seen to be $\frac{\pi}{2a^4}$.

Thus $$\int_0^\infty \frac{\sin rx}{x(a^2+x^2)}dx=\frac{\pi}{4a^4}\left\{2-e^{-ar}(ar+2)\right\},$$

whence putting $a=1$, the first result is obtained.

Next, consider the identity

$$\frac{\sin^2 rx}{x^2(1+x^2)}=\left(\frac{1}{x^2}-\frac{1}{1+x^2}\right)\sin^2 rx=\frac{\sin^2 rx}{x^2}+\frac{\cos 2\,rx}{2(1+x^2)}-\frac{1}{2(1+x^2)}.$$

The second integral is thus equal to

$$\int_0^\infty \frac{\sin^2 rx}{x^2}dx+\tfrac{1}{2}\int_0^\infty \frac{\cos 2\,rx}{1+x^2}dx-\tfrac{1}{2}\int_0^\infty \frac{dx}{1+x^2}.$$

$$=\frac{\pi r}{2}+\frac{1}{2}\frac{\pi}{2}e^{-2r}-\frac{1}{2}\frac{\pi}{2}.$$ (Gibson, p. 455.)

$$=\frac{\pi}{4}(2r+e^{-r}-1).$$

(2) We know $\int_0^\infty \frac{\cos rx\,dx}{(1+x^2)^2}=\frac{\pi}{4}e^{-r}(1+r)$. (J. I. M. S. 1913, p. 230.)

Integrating with respect to r,

$$\int_0^\infty \frac{\sin rx}{x(1+x^2)^2}dx=-\frac{\pi}{4}e^{-r}(r+2)+C.$$

To find C, put $r=0$; we get $C=\frac{\pi}{2}$.

Hence the first integral $=\frac{\pi}{4}\left\{2-e^{-r}(r+2)\right\}$.

Again $\int_0^\infty \frac{\sin 2\,rx}{x(1+x^2)}dx=\frac{\pi}{2}(1-e^{-2r})$. (Gibson, p. 469).

Integrating with respect to r

$$\int_0^\infty \frac{\sin^2 rx}{x^2(1+x^2)}dx=\frac{\pi}{2}(r+\tfrac{1}{2}\,e^{-2r})+C.$$

Put $r=0$, then $C=-\frac{\pi}{4}$.

Hence the second integral$=\frac{\pi}{4}(-1+e^{-2r}+2r.)$

Question 498.

(R. N. Apte, M.A., F.R.A.S.) :—The parabola $y=ax^2+bx+c$ is drawn at random (where a, b, c, are real). Show that the probability of its cutting the x axis in imaginary points is $\frac{31}{72}-\frac{1}{12}\log 2$.

Solution (1) *by K. J. Sanjana M.A.*, (2) *by T. P. Trivedi M.A., L.L.B.*

(1) The x—axis being $y=o$, we get $ax^2+bx+c=0$; so that we have to find when the roots of this equation are imaginary. This requires $b^2<4ac$, which is impossible if a and c have opposite signs; and the chance that a and c have the same sign is $\frac{1}{2}$. Hence if, under this condition, the chance of $b^2<4ac$ is p, the probability required will be $\frac{1}{2}p$, when a and c are both taken positive.

The chance that one of the three a, b, c, is greater than the other two is $\frac{1}{3}$. The values of x depend only on the ratios of the coefficients, so that we can divide out by any one of these: if we divide out by the greatest, the other two are each made to lie between 0 and 1.

Let a be the greatest: the equation reduces to $x^2+bx+c=0$, with the condition $b^2<4c$. Thus $c>\frac{b^2}{4}$ which is less than unity; hence the limits of c are 1 and $\frac{b^2}{4}$, those of b being 1 and 0. The chance in this case$=\int_0^1 db\int_0^1 dc=\left\{b-\frac{b^3}{12}\right\}_0^1=\frac{11}{12}$. Similarly, if c is the greatest, the chance for imaginary roots is $\frac{11}{12}$.

Let b be the greatest, the equation now reduces to $ax^2+x+c=0$, with $1<4ac$. Thus $c>\frac{1}{4a}$, so that its limits are 1 and $\frac{1}{4a}$. The limits of a are no longer 1 and 0, for if $0<a<\frac{1}{4}$, $c>1$ which is impossible. Thus the chance in this case is

$$\int_{\frac{1}{4}}^{1} da\int_{\frac{1}{4a}}^{1} dc=\left\{a-\tfrac{1}{4}\log a\right\}_{\frac{1}{4}}^{1}=1-\left(\tfrac{1}{4}-\tfrac{1}{4}\log\tfrac{1}{4}\right)=\tfrac{3}{4}-\tfrac{1}{2}\log 2.$$

Therefore finally, the required chance is

$$\tfrac{1}{2}\cdot\tfrac{1}{3}\left\{\frac{11}{12}+\frac{11}{12}+\frac{9}{12}-\tfrac{1}{2}\log 2\right\}=\frac{31}{72}-\frac{1}{12}\log 2.$$

(2) We have to find the probability that the roots of the equation $ax^2+bx+c=0$ are imaginary. We may suppose for the sake of convenience that a, b, c all lie between 1 and -1 as the equation may be divided out by any suitable number however large. The roots are imaginary if a and c have the same sign and $ac>\frac{b^2}{4}$.

The probability that a and c have the same sign is $\frac{1}{2}$. Let a and c be both positive and suppose p is the probability for imaginary roots; then p is also the probability for imaginary roots when a and c are both negative. Therefore p is the probability for the roots being imaginary when a and c have the same sign whether positive or negative and the *absolute* probability for imaginary roots $=\frac{1}{2}p$. To find p we observe that ac being $>\frac{b^2}{4}$, a may have any value from $\frac{b^2}{4}$ to 1 and then the values of c range from $\frac{b^2}{4a}$ to 1; also b lies between 1 and -1.

Hence
$$p=\frac{\int_{-1}^{1}\int_{\frac{1}{4}b^2}^{-1}\int_{\frac{1}{4}b^2}^{1} db\,da\,dc}{\int_{-1}^{1}\int_{0}^{1}\int_{0}^{1} db\,da\,dc};$$

the quantities a and c need only be taken between 1 and 0, since we confine our attention to positive values only.

But
$$\int_{-1}^{1}\int_{\frac{1}{4}b^2}^{1}\int_{\frac{1}{4}b^2}^{1} db\,da\,dc=\int_{-1}^{1}\int_{\frac{1}{4}b^2}^{1}\left(1-\frac{b^2}{4}\right)db\,da$$

$$=\int_{-1}^{1}\left\{1-\frac{b^2}{4}+\frac{b^2}{4}\log\frac{b^2}{4}\right\}db$$

$$=2\int_{0}^{1}\left\{1-\frac{b^2}{4}+\frac{b^2}{2}\log\frac{b}{2}\right\}db=\frac{31}{18}+\frac{1}{3}\log\tfrac{1}{2};$$

and
$$\int_{-1}^{1}\int_{0}^{1}\int_{0}^{1} db\,da\,dc=2.$$

Thus $p=\frac{31}{36}+\frac{1}{6}\log\frac{1}{2}$, and the absolute probability is $\frac{1}{2}p=\frac{31}{72}-\frac{1}{12}\log 2$.

Question 499.

(K. J. SANJANA) :—A chord AKB of a curve is drawn perpendicular to the normal PK at any point P of the curve, and M is its mid-point; prove that when PK is infinitesimal the ratio of the radius of curvature at P to the radius of curvature at the corresponding point of the evolute is PK: 3 KM. (Suggested by Question 470.)

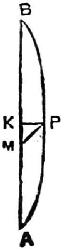
Solutions (1) *by T. P. Trivedi, M.A. L.L.B*; (2) *by T.S. Krishna Rao.*

(1) Let ρ be the radius of curvature at P on the curve and ρ' the the radius of curvature at the corresponding point of the evolute.

By Question 470 (J.I.M.S.) $\frac{\rho'}{\rho}=\frac{3y_1\, y_2^2-y_3\,(1+y_1^2)}{y_2^2}$.

Also if the angle KPM be θ; then

$\tan\theta=\left\{ y_1-\frac{(1+y_1^2)y_3}{3y_2^2}\right\}$. [Edward's *Diff. Calc.* Ex. 38, page. 110.]



$$\therefore \quad \frac{\text{KM}}{\text{PK}}=\frac{3y_1\, y_2^2-y_3\,(1+y_1^2)}{3y_2^2}=\frac{\rho'}{3\rho};$$

$$\therefore \quad \frac{\rho}{\rho'}=\frac{\text{PK}}{3\text{ KM}}.$$

(2) Referred to the tangent and normal at the point as axes, the equation of the curve may be written in the form,

$$y=ax^2+2hxy+by^2+\text{higher powers of } x \text{ and } y. \quad \ldots \quad (1)$$

The equation of the chord is $y=k$, say.

The points in which $y=k$ cuts the curve are given by

$$k=ax^2+2hkx+bk^2,$$

if we neglect higher powers of x and y than the second.

$$\therefore \quad x_1+x_2=-\frac{2hk}{a}.$$

Hence, the co-ordinates of the mid-point of the chord, are $(-hk/a, k)$

Now, by Q. 470, the ratio of the radius of curvature at P to that at the corresponding point of the evolute is given by

$$\frac{\rho'}{\rho}=\frac{3y_1\, y_2^2-y_3\,(1+y_1^2)}{y_2^2}.$$

To find y_1, y_2, y_3 at the origin, differentiate (1) successively;

$$y_1 = 2ax + 2hy + 2hxy_1 + 2byy_1$$
$$y_2 = 2a + 2hy_1 + 2hxy_2 + 2hy_1 + 2byy^2 + 2by_1^2$$
$$y_3 = 2hy_2 + 2hxy_3 + 2hy_2 + 2hy_2 + 2byy_3 + 2by_1y_2 + 4by_1y_2$$

Thus $(y_1)_0 = 0$, $(y_2)_0 = 2a$, $(y_3)_0 = 12\,ah$,

and the ratio $$\frac{\rho'}{\rho} = \frac{-12ah}{4a^2} = -\frac{3h}{a}.$$

But $$\text{KM} = -\frac{hk}{a}, \text{ and PK} = k.$$

Hence $$\frac{\text{PK}}{3\text{ KM}} = -\frac{a}{3h} = \frac{\rho}{\rho'}.$$

Question 500.

(V. K. ARAVAMUDAN) :—Solve the equation

$$\frac{dy}{dx} + y^2 + \frac{2y}{x} + x^2 - \frac{2}{x^2} = 0.$$

Solution by K. J. Sanjana, M.A., and T. P. Trivedi, M.A., LL.B.

Putting $y = \frac{1}{u}\frac{du}{dx}$ and simplifying, we get

$$x^2\frac{d^2u}{dx^2} + 2x\frac{du}{dx} + (x^4 - 2)u = 0.$$

To solve this we take $u = a_1x^m + a_2x^{m+r} + a_3x^{m+s} + \ldots$, where m is the lowest exponent of x and r, s, ...are positive quantities in ascending order. The substitution gives

$$a_1x^m\{m(m-1) + 2m - 2\} + a_2x^{m+r}\{(m+r)(m+r-1) + 2(m+r) - 2\}$$
$$+ a_3x^{m+s}\{(m+s)(m+s-1) + 2(m+s) - 2\} + \ldots\ldots$$
$$+ a_1x^{m+4} + a_2x^{m+4+r} + a_3x^{m+4+s} + \ldots = 0.$$

As $a_1 \neq 0$, we get $m^2 + m - 2 = 0$, giving $m = 1$ or -2. A little consideration shows that r must be equal to 4; for, if it were 1, 2, or 3, a_2 would be zero and the term in x^{m+r} would be wanting; so also s, t,...are 8, 12,......Thus we have

$$a_2\{(m+r)(m+r-1) + 2(m+r) - 2\} + a_1 = 0;$$

and more generally

$$a_{k+1}\{(m+kr)(m+kr-1) + 2(m+kr) - 2\} + a_k = 0,$$

whence $a_{k+1} = -a_k \div \{(m+kr+2)(m+kr-1)\}$.

When $m=1$, $a_2=-\frac{a_1}{4\cdot7}$, $a_3=-\frac{a_2}{8\cdot11}=\frac{a_1}{4\cdot7\cdot8\cdot11}$, $a_4=-\frac{a_1}{4\cdot7\cdot8\cdot11\cdot12\cdot15}$,

and so on. This gives $u=a_0\left\{x-\frac{x^5}{4\cdot7}+\frac{x^9}{4\cdot7\cdot8\cdot11}-\dots\right\}=a_0u_1$, suppose.

Similarly, when $m=-2$ we get $b_0u_2=b_0\left\{x^{-2}-\frac{x^2}{1\cdot4}+\frac{x^6}{1\cdot4\cdot5\cdot8}-\dots\right\}$:

so that x is seen to be $a_0u_1+b_0u_2$.

Hence $y=\frac{du}{dx}\div u=\frac{a_0u'_1+b^0u'_2}{a_0u_1+b_0u_2}$, where the dashes denote differentials

with regard to x. This may finally be written

$$y=\frac{cu_1'+u_2'}{cu_1+u_2},$$

where $c\equiv\frac{a_0}{b_0}$ is the single constant required by the proposed equation of the first order.

Question 501.

(K. J. Sanjana, M.A.) :—If $(r, n)=1^r+2^r\dots+n^r$, shew that $(r, n)-{}_{r+1}C_1(r, n-1)+{}_{r+1}C_2(r, n-2)-\dots(-1)^{r+1}{}_{r+1}C_{r+1}(r, n-r-1)=r\,!$ where n is any integer greater than $r+1$. When $n=r+1$, or r, prove that the formula is still true, if the last or the last two terms are neglected; and find the value of the series when $n=r-1$.

Solution by R. Vythynathaswamy.

$$r\,!=\Delta^r(n-r)=n^r-{}_rC_1(n-1)^r+{}_rC_2(n-2)^r\dots+{}_rC_r(n-r)^r \qquad \dots \quad (1)$$

Also, the left-hand side of the question proposed may be written as

$$\begin{aligned}
&n^r-{}_rC_1(n-1)^r+{}_rC_2(n-2)^r\dots\dots\dots\pm{}_rC_r(n-r)^r.\\
&\quad-{}_rC_0(n-1)^r+{}_rC_1(n-2)^r\dots\dots\dots\dots\pm{}_rC_r(n-r-1)^r\\
&+(n-1)^r-{}_rC_1(n-2)^r+{}_rC_2(n-3^r\dots\dots\\
&\quad-{}_rC_0(n-2)^r+{}_rC_1(n-3)^r\dots\dots\dots\\
&\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\\
&+2^r-{}_rC_1.1^r.\\
&\qquad-{}_rC_0\cdot1^r\\
&+1^r.
\end{aligned}$$

Hence the left-hand side $=n^r-{}_rC_1(n-1)^r+{}_rC_2(n-2)^r\dots+r\,!$ by (1).

In order that all the terms in the given series may be significant, $n>r+1$,

If $n=r+1$, or, $<r+1$, some of the terms at the end become zero, and some impossible.

When $n=r-1$, the last three terms are not significant.

Question 504.

(R. SRINIVASAN, M.A.) :—Shew that

$$\int_0^{\pi} \theta \sin\theta \cos^{2m}\theta \cos(a\cos\theta)d\theta=\frac{\pi}{a^{2m+1}}\left\{\text{P}\sin a+\text{Q}\cos a\right\},$$

where $\text{P}=a^{2m}-2m(2m-1)a^{2m-2}+2m(2m-1)(2m-2)(2m-3)a^{2m-4}-\ldots$

$\text{Q}=2ma^{2m-1}-2m(2m-1)(2m-2)a^{2m-3}+\ldots\ldots\ldots\ldots$

Solution (1) *by T. P. Trivedi, M.A., L.L.B. and K. J. Sanjana, M.A.* (2) *by R. Vythyanathaswamy.*

(1) Let $\text{I}=\int_0^{\pi} \theta\sin\theta\cos^{2m}\theta\cos(a\cos\theta)d\theta$

$$=\int_0^{\pi}(\pi-\theta)\sin\theta\cos^{2m}\theta\cos(a\cos\theta)d\theta.$$

$$\therefore \quad 2\text{I}=\pi\int_0^{\pi}\sin\theta\cos^{2m}\theta\cos(a\cos\theta)\,d\theta$$

$$=\pi\left\{\left[-\frac{\sin(a\cos\theta)}{a}\cos^{2m}\theta\right]_0^{\pi}-\frac{1}{a}\int_0^{\pi}2m\cos^{2m-1}\theta\sin\theta\sin(a\cos\theta)\,d\theta\right\}$$

$$=\pi\left\{\frac{2\sin a}{a}-\frac{2m}{a}\int_0^{\pi}\cos^{2m-1}\theta\sin\theta\sin(a\cos\theta)d\theta\right\}$$

$$=\pi\left\{\frac{2\sin a}{a}-\left[\frac{2m}{a}\frac{\cos^{2m-1}\theta\cos(a\cos\theta)}{a}\right]_0^{\pi}-\frac{2m(2m-1)}{a^2}\int_0^{\pi}\cos^{2m-2}\theta\sin\theta\cos(a\cos\theta)\,d\theta\right\}$$

$$=\pi\left\{\frac{2\sin a}{a}+\frac{2\times 2m\cos a}{a^2}\,.\,-\frac{2m(2m-1)}{a^2}\times\int_0^{\pi}\cos^{2m-2}\theta\sin\theta\cos(a\cos\theta)d\theta\right\}$$

$$=2\pi\left\{\frac{\sin a}{a}+\frac{2m\cos a}{a^2}-\frac{2m(2m-1)}{a^3}\sin a-\frac{2m(2m-1)(2m-2)}{a^4}\cos a+\ldots\right\}$$

$$\therefore \qquad I=\frac{\pi}{a^{2m+1}}\Big\{P \sin a+Q \cos a\Big\},$$

where P and Q have the values stated in the question.

(2) We have $\int_{-1}^{+1} \sin mx\, dx = 0.$

Differentiating with respect to m successively

$$\int_{-1}^{+1} x^{2r+1} \cos mx dx + 0.$$

Thus $\int_{-1}^{+1} \cos mx \sin^{-1}(nx) dx = 0,$

Again

$$\int_{-1}^{+1} \cos mx \cos^{-1}(nx) dx = \pi \frac{\sin m}{m},$$

putting $\cos^{-1}nx, = \frac{\pi}{2} - \sin^{-1}nx$, and expanding the inverse function in terms of odd powers of x.

Now, consider the integral

$$I = \int_0^{\pi} \theta \sin\theta \cos(\alpha \cos\theta)\, d\theta.$$

This is the same as $\int_{-1}^{+1} \cos^{-1}x \cos(\alpha x) dx$, where $x = \cos\theta$.

$$\therefore \qquad I = \frac{\pi \sin\alpha}{\alpha} \text{ by (i).}$$

Differentiate this result $2m$ times with respect to α, and we get

$$(-)^m \int_0^{\pi} \theta \sin\theta \cos(\alpha\cos\theta) \cos^{2m}\theta\, d\theta = \pi \frac{d^{2m}}{d\alpha^{2m}} \left(\frac{\sin\alpha}{\alpha}\right).$$

By Leibnitz' Theorem the right hand side is

$$\pi \Big\{ (-)^m \sin\alpha \cdot \frac{1}{\alpha} + 2m\, C_1 \cdot (-)^m \cos\alpha \cdot \frac{\lfloor 1}{\alpha^2}$$

$$+ 2m\, C_2 (-)^{m-1} \sin\alpha \cdot \frac{\lfloor 2}{\alpha^3} \ldots \qquad \ldots + \frac{\lfloor 2m}{\alpha^{2m+1}} \Big\}.$$

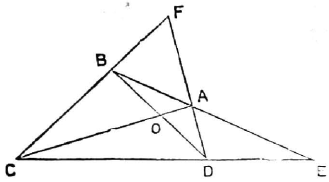
Hence, removing $(-)^m$, we have

$$I = \frac{\pi}{\alpha^{2m+1}} (P \sin\alpha + Q \cos\alpha),$$

where P and Q have the values given.

Question 506.

(A. C. L. WILKINSON, M.A., F.R.A.S.) :—Show that it is impossible to describe a plane quadrilateral such that the angle between the diagonals is equal to each of the angles between the two pairs of opposite sides.



Solution by K. J. Sanjana, M. A. and several others.

The angle AOD may be equal to the angle F; but it is>the angle E, as it is>BDC, which itself is>E. Similarly, AOB may be equal to E, but must be>F. Hence the result stated.

Question 507.

(S. RAMANUJAN) :—Solve completely $x^2=y+a$, $y^2=z+a$, $z^2=x+a$, and hence shew that

(a) $\sqrt{8-\sqrt{8+\sqrt{8-\text{etc}}}} = 1+2\sqrt{3}\sin 20^\circ$

(b) $\sqrt{11-2\sqrt{11+2\sqrt{11-\text{etc}}}} = 1+4\sin 10^\circ$

(c) $\sqrt{23-\sqrt{23+2\sqrt{23+2\sqrt{23-2\sqrt{23+\text{etc}}}}}}$

$$=1+4\sqrt{3}\sin 20^\circ.$$

Solution by the Proposer.

We have,

$$x=z^2-a=(y^2-a)^2-a=\{(x^2-a)^2-a\}^2-a.$$

or $\quad x^8-4x^6a+2x^4(3a^2-a)-4x^2(a^3-a^2)-x+a^4-2a^3+a^2-a=0 \ldots \quad (1)$

Evidently y and z also are the roots of this equation.

But x may be written as

$$\sqrt{a+y}=\sqrt{a+\sqrt{a+z}}$$

$$=\sqrt{a+\sqrt{a+\sqrt{a+x}}}=$$

$$\sqrt{a+\sqrt{a+\sqrt{a+\&c}}}=\sqrt{a+x}.$$

Hence we have $x^2-x-a=0$ ... ... (2)

Therefore x^2-x-a must be a factor of the expression in (1).

Now, dividing (1) by x^2-x-a, we have

$$x^6+x^5+x^4(1-3a)+x^3(1-2a)+x^2(1-3a+3a^2)$$
$$+x(1-2a+a^2)+(1-a+2a^2-a^3)=0. \quad ... \quad (3)$$

Let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, be the roots, of this equation.

Then since y and z are also roots we may suppose

that $\left.\begin{array}{l}\alpha^2=a+\beta\\ \beta^2=a+\gamma\\ \gamma^2=a+\alpha\end{array}\right\}$ $\left.\begin{array}{l}\alpha'^2=a+\beta'\\ \beta'^2=a+\gamma'\\ \gamma'^2=a+\alpha'\end{array}\right\}$

Let $\alpha+\beta+\gamma=u$ and $\alpha'+\beta'+\gamma'=v$, then we see that

$$\alpha^2+\beta^2+\gamma^2=3a+u,$$

that is $$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{u^2-u-3a}{2}.$$

Again we have,

$\left.\begin{array}{r}\alpha^2\beta=a\beta+\beta^2\\ \beta^2\gamma=a\gamma+\gamma^2\\ \text{and } \gamma^2\alpha=a\alpha+\alpha^2\end{array}\right\}$ and $\left.\begin{array}{l}\gamma\alpha^2=a\gamma+\beta\gamma\\ \alpha\beta^2=a\alpha+\gamma\alpha\\ \beta\gamma^2=a\beta+\alpha\beta\end{array}\right\}.$

Adding up all the six results we have

$$\Sigma\alpha^2(\beta+\gamma)=2a(\alpha+\beta+\gamma)+$$
$$\alpha^2+\beta^2+\gamma^2+\alpha\beta+\beta\gamma+\gamma\alpha\,;$$

i.e., $$(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma=$$
$$2a(\alpha+\beta+\gamma)+(\alpha^2+\beta^2+\gamma^2)+(\alpha\beta+\beta\gamma+\alpha\gamma).$$

i.e., $$u.\ \tfrac{1}{2}(u^2-u-3a)-3\alpha\beta\gamma=2au+(3a+u)+\tfrac{1}{2}(u^2-u-3)$$

Hence we have $\alpha\beta\gamma=u^3-2u^2-7au-u-3a.$

Similarly we have

$$\alpha'+\beta'+\gamma'=v$$
$$\alpha'\beta'+\beta'\gamma'+\gamma'\alpha'=\tfrac{1}{2}(v^2-v-3a)$$
$$6\ \alpha'\beta'\gamma'=v^3-2v^2-7av-v-3a.$$

Hence the sextic in (3) is identical with

$$\{x^3-x^2u+\tfrac{1}{2}x(u^2-u-3a)-\tfrac{1}{6}(u^3-2u^2-7au-u-3a)\}$$
$$\times\{x^3-x^2v+\tfrac{1}{2}x(v^2-v-3a)-\tfrac{1}{6}(v^3-2v^2-7av-v-3a)\} \quad ... \quad (4)$$

Now equating the co efficients of x^5 as well as x^3 in (3) and (4) we have

$$u+v=-1$$
$$u^3+v^3-2(u^2+v^2)-(7a+1)(u+v)-6a+$$
$$3\{uv(u+v)-2uv-3a(u+v)\}=6(2a-1).$$

Substituting for $u+v$ in the above result, we have, $uv=2-a$.

Hence
$$u=-\frac{1+\sqrt{4a-7}}{2}$$
and
$$v=-\frac{1-\sqrt{4a-7}}{2} \quad \ldots \quad \ldots (5)$$

These values of u and v substituted in (4) reduce the sextic equation in (3) to the two cubics :

$$x^3+x^2\frac{1-\sqrt{4a-7}}{2}-x\frac{2a+1+\sqrt{4a-7}}{2}+\frac{a-2+a\sqrt{4a-7}}{2}=0,$$

$$x^3+x^2\frac{1+\sqrt{4a-7}}{2}-x\frac{2a+1-\sqrt{4a-7}}{2}+\frac{a-2-a\sqrt{4a-7}}{2}=0,$$

which can be solved by the usual methods.

In the numerical examples proposed, the combinations of the signs *plus* and *minus* may be determined by proceeding as follows :—

$$1+2\sqrt{3}\sin 20^\circ=\sqrt{1+4\sqrt{3}\sin 20^\circ+12\sin^2 20^\circ}$$

$$=\sqrt{7+4\sqrt{3}\sin 20^\circ-6\cos 40^\circ}$$

$$=\sqrt{7+4\sqrt{3}\sin 20^\circ-4\sqrt{3}\cos 30^\circ\cos 50^\circ}$$

$$=\sqrt{7+4\sqrt{3}\sin 20^\circ-2\sqrt{3}\cos 70^\circ-2\sqrt{3}\cos 10^\circ}$$

$$=\sqrt{7+2\sqrt{3}\cos 70^\circ-2\sqrt{3}\cos 10^\circ}$$

$$=\sqrt{7-4\sqrt{3}\sin 30^\circ\sin 40^\circ}=\sqrt{8-(1+2\sqrt{3}\sin 40^\circ)}$$

$$=\sqrt{8-\sqrt{8+(2\sqrt{3}\sin 80^\circ-1)}}$$

$$=\sqrt{8-\sqrt{8+\sqrt{8-(1+2\sqrt{3}\sin 20^\circ)}}}.$$

In a similaar manner we have,

$$1+4\sin 10^\circ=\sqrt{11-2(1+4\sin 50^\circ)}$$

$$=\sqrt{11-2\sqrt{11+2(4\sin 70^\circ-1)}}$$

$$=\sqrt{11-2\sqrt{11+2\sqrt{11-2(1+4\sin 10^\circ)}}},$$

and also

$$1+4\sqrt{3}\sin 20^\circ=\sqrt{23-2(4\sqrt{3}\sin 80^\circ-1)}$$

$$=\sqrt{23-2\sqrt{23+2(1+4\sqrt{3}\sin 40^\circ)}}$$

$$=\sqrt{23-2\sqrt{23+2\sqrt{23+2(1+4\sqrt{3}\sin 20^\circ)}}}.$$

QUESTIONS FOR SOLUTION.

532. (R. VYTHYNATHASWAMY) :—If 2A, 2B, 2C are the mutual inclinations of any three lines in space and 2L, 2M, 2N, the angles which any fourth straight line makes with them, shew that the following identical relation holds

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 4 \\ 1 & 0 & \sin^2 C & \sin^2 B & \sin^2 L & 1 \\ 1 & \sin^2 C & 0 & \sin^2 A & \sin^2 M & 1 \\ 1 & \sin^2 B & \sin^2 A & 0 & \sin^2 N & 1 \\ 1 & \sin^2 L & \sin^2 M & \sin^2 N & 0 & 1 \\ 4 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

533. (PROF. K. J, SANJANA.)—Three real numbers a, b, c, are written down at random. Find the chance (i) that $b^2 < 4ac$, (ii) that $b^2 > 4ac$. What is the chance of b^2 being equal to $4ac$?

534. (ZERO) :—A given ellipse is projected orthogonally on a given plane. Shew how to obtain the axes of the projection geometrically.

535. (N. P. PANDYA) :—Construct a conic having given three points on it and the symmedian point and orthocentre of the triangle formed by the tangents at those points.

536. (K. APPUKUTTAN ERADY, M. A.) :—Prove that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u e^{-u'}\, dx\, dy = \frac{\pi}{2}\,\frac{(ab'+a'b-2hh')}{(a'b'-h'^2)^{\frac{3}{2}}},$$

where $u \equiv ax^2+2hxy+by^2$ and $u' \equiv a'x'+2h^1xy+by^2$.

537. (K. J. SANJANA, M.A.) :—Prove that

$$\frac{1}{2}\,\frac{x^2}{2} - \frac{1\cdot3}{2\cdot4}\,\frac{x^4}{4} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{x^6}{6} - \dots\dots$$

$$= \tfrac{1}{4}\log(1+x^2) + \tfrac{1}{8}\log(1+x_1^2) + \tfrac{1}{16}\log(1+x_2^2) + \dots,$$

where $x_1^2 = \dfrac{x^4}{4(x^2+1)}$, $x_2^2 = \dfrac{x_1^4}{4(x_1^2+1)}$, ... ;

and that $\frac{1}{2}\frac{x^2}{2}+\frac{1\cdot3}{2\cdot4}\frac{x^4}{4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^6}{6}+\quad\dots$

$$=-\tfrac{1}{4}\log(1-x^2)-\tfrac{1}{8}\log(1+x_1^2)-\tfrac{1}{16}\log(1+x_2^2)-\dots$$

where $x_1^2=\frac{x^4}{4(1-x^2)},\ x_2^2=\frac{x_1^4}{4(1+x_1^2)},\ \dots\dots\dots$

In the first equality x is any real quantity; in the second, any real quantity <1.

538. (A. NARASINGA RAO) :—If the tangents at the circular points of a nodal circular cubic curve meet at P on the curve, then any transversal through P cuts the curve again in pairs of points such that the lines joining them to the node form a pencil in orthogonal involution.

539. (S. KRISHNASWAMI AIYANGAR) :—The focal chords of the maximum inscribed ellipse of the triangle of reference parallel to the sides are the roots of the equation

$$x^3-2x^2\Delta^{\frac{1}{2}}\tan^{\frac{1}{2}}\lambda\cot w+x\Delta\tan\lambda\operatorname{cosec}^2 w-2R^2\Delta^{\frac{1}{2}}\tan^{\frac{3}{2}}\lambda=0,$$

where λ is the minor semi-steiner angle, w is the Brocard angle and R the circumradius.

540. (S. KRISHNASWAMI AIYANGAR) :—If S, H be the foci of the maximum inscribed ellipse of ABC and SH subtends angles θ_1, θ_2, θ_3 at the vertices, prove that

$$bc\cot\theta_1+ca\cot\theta_2+ab\cot\theta_3=\Sigma(a^2)-\{\Sigma(a^4)-\Sigma(a^2b^2)\}^{\frac{1}{2}}.$$

541. (S. RAMANUJAN) :—Prove that

$$1+\frac{1}{1\cdot3}+\frac{1}{1\cdot3\cdot5}+\frac{1}{1\cdot3\cdot5\cdot7}+\dots+\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{4}{1+}\dots=\sqrt{\left(\frac{\pi e}{2}\right)}$$

542. (K. V. ANANTANARAYANA SASTRI, B.A.) :—Expand $\theta\cot\frac{1}{4}\theta$ in powers of $\cos\theta$.

543. (R. VYTHYNATHASWAMY) :—Shew that

$$1^r+2^r+3^r+\dots\dots n^r=(c_2\Delta+c_3\Delta^2+\dots\dots c_{r+1}\Delta^r)0^r,$$

where $c_p={}_{n+1}C_p$.

544. (SELECTED) :—Evaluate the definite integrals

$$\int_0^\infty e^{-x^2}\sin ax^2dx\,;\ \int_0^\infty e^{-x^2}\cos ax^2dx,$$

545. (M. T. NARANIENGAR, M.A.) :—Shew that the roots of $8x^3-18x+9=0$, and $8y^3-18y-9=0$, are connected by the relations

$$x=3+2y-2y^2\text{ ; and } y=3-x-2x^2.$$

546. (S. RAMANUJAN) :—Shew that

$$\text{(i)}\quad \left(\frac{1}{3}-\frac{1}{4}\right)+\frac{2}{3^2}\left(\frac{1}{3}-\frac{1}{4^2}\right)+\frac{2\cdot4}{3\cdot5^2}\left(\frac{1}{3}-\frac{1}{4^3}\right)+\dots\dots\dots$$

$$=\frac{\pi}{12}\log(2+\sqrt{3})$$

$$\text{(ii)}\quad 1-\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}+\dots\dots\dots=\frac{\pi^2}{8}-\tfrac{1}{2}\{\log(1+\sqrt{2})\}^2$$

547. (V. K. ARAVAMUDAN :—Prove that the oblique trajectories of the system of curves represented by the elimination of t between

$$x=f(t,a),\ y=\phi(t,a)$$

a being the parameter of the family, are given by replacing a by the complete integral of

$$n\left\{\left(\frac{\partial f}{\partial t}\right)^2+\left(\frac{\partial\phi}{\partial t}\right)^2\right\}+\frac{\partial a}{\partial t}\left\{n\left[\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial a}+\frac{\partial f}{\partial t}\frac{\partial f}{\partial a}\right]\right.$$

$$\left.-\frac{\partial\phi}{\partial t}\frac{\partial f}{\partial a}+\frac{\partial f}{\partial t}\frac{\partial\phi}{\partial a}\right\}=0$$

and hence find the trajectories of a system of confocal ellipses, n being equal to the tangent of the angle of intersection.